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Introducing Immanent Structuralism

1.1 Introduction

The aim of this dissertation will be to defend a specific form of structuralism in the philosophy of mathematics, which I call "immanent structuralism." Structuralism holds that mathematics is about structures or patterns. *Immanent* structuralism holds that these structures or patterns are universals or properties that can be (but need not always be) literally instantiated by many different kinds of things, in particular physical systems. Immanent structuralism is distinct from standard, *ante rem* structuralism, in that it holds structures to be what are called purely structural universals, rather than systems consisting of a special sort of intrinsically featureless object or particular.[1] According to immanent structuralism, a true mathematical statement holds iff – and because – certain facts about the natures of purely structural properties obtain.

1.2 Platonism and Ante Rem Structuralism

Structuralism can be illustrated by contrast with traditional Platonism. According to traditional Platonists, mathematical objects are *sui generis*. They are their own fundamental kind of entity. Moreover, for the Platonist, some mathematical objects are particulars. For instance, the number two is to be thought of as an abstract individual, i.e., an object, or a particular thing.

Platonism faces several challenges:

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[1] Cf. *ante rem* theorist Michael Resnik's (1997), p. 201: "The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures. And as such they have no identity or distinguishing features outside a structure."

- **The Epistemological Challenge**: How can most people have reliable beliefs about mathematics if that requires reliable beliefs about causally inert abstract objects?[2]

- **The Ontological Challenge**: The fewer the fundamental kinds of entities one posits, the better. So a theory that can plausibly amend mathematical ontology to some already-recognized category is preferable.

- **The Applicability Challenge**: How can we learn things about the physical world by mathematics (and not just in physics, but in all sorts of natural sciences) if mathematics is about a realm of non-physical entities?

In addition, *reductive* versions of Platonism, which try to reduce most classes of mathematical object to some subset of them, face a serious problem regarding the nature of mathematical reduction. For instance, consider a "naïve" set-theoretic Platonism, according to which all mathematical objects *just are* sets.[3] On this view, sets are all we need *ontologically speaking* to make sense of mathematics.

One challenge for this view is that mathematical practice seems to allow for multiple, equally salient reductions of the natural numbers to sets. For instance, consider the following proposed reduction from Von Neumann. Let us call the following series of sets the "V-Sets":

- **(V-Sets):** 0: {}, 1:{{}}, 2: {{},{{}}}, … [where the n+1th set is the power set of the nth set]

Consider also the following series, the "Z-Sets" (due to Zermelo):

- **(Z-Sets):** 0: {}, 1: {{}}, 2: {{{}}}, … [where the n+1th set is the set of the nth set]

The problem for naïve set-theoretic Platonism is that the Z-Sets are just as good for a mathematical reduction of arithmetic to set theory as are the V-Sets. Thus, reductive, set-theoretic Platonism faces an additional problem:

- **The Multiple-Reductions Problem**: There are multiple, equally good possible reductions of the ontology of numbers to the ontology of sets.

This last problem has inspired the structuralist view, which treats mathematics as the science of *structures*, or *patterns*.

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[2] Benacerraf (1973)

[3] So on this "reductive" view, sets are the only kind of *sui generis* abstract mathematical object. Set-theoretic Platonism is probably the most common form of reductive Platonism.

According to the standard version of structuralism advanced by Resnik (1997) and Shapiro (1997) – *ante rem* structuralism – the subject matter of mathematics consists of "an ontology of featureless objects, called 'positions', and … systems of relations or 'patterns' in which these positions figure."[4] Ante rem structuralists view individual mathematical objects as the "nodes" or "positions" within these systems.[5] For example, the *natural number system* is a structure of intrinsically featureless objects with the order characteristic of the natural numbers.[6] Thus, structuralists will say that the V-sets and the Z-sets both *exemplify* the natural number pattern, but that neither is, strictly speaking, *identical* with the series of natural numbers.[7]

On this view, the natures of mathematical objects (such as the number two, say) are wholly constituted by the relations they bear to other positions within the structure.[8] In other words, the intrinsic natures of mathematical objects are exhausted by their relations to the other objects in the larger structure they are a part of.

However, ante rem structuralism, which conceives of structures and their positions as abstract objects, still seems to suffer from the three problems of traditional Platonism, viz., the epistemological, ontological, and applicability problems. Additionally, ante rem structuralism takes on a seemingly more obscure ontology than Platonism, in that it is committed to objects that are not only abstract, but whose natures are entirely exhausted by their relations to other such objects. While I think there is *something* to this idea, it would be better if we did not have to expand our ontology in this way to include this apparently esoteric sort of object with such an unusual nature. I will say a bit more about these issues in Chapter 5.

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[4] Resnik (1997) p. 269.

[5] Note that, like the Platonist, ante rem structuralists interpret reference to mathematical objects as straightforwardly singular and referential. See Shapiro (1997) pp. 10-11. See also p. 13: "According to ante rem structuralism, the variables of the theory range over the places of that structure, the singular terms denote places in that structure, and the relation symbols denote the relations of the structure." And p. 83: "Places in structures are bona fide objects. … Bona fide singular terms … like "2" denote bona fide objects."

[6] I.e., they constitute an omega-series.

[7] Note that, on the ante rem structuralist view, "The natural-number structure *itself* exemplifies the natural number structure." (Shapiro 1997, p. 101, emphasis added)

[8] Resnik: "In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics … are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure." Shapiro: "The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other."

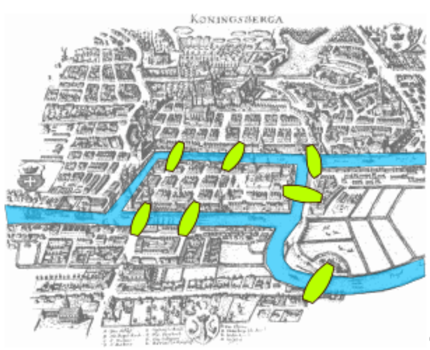
1.3 Structural Universals

Faced with these issues, let us look at the version of structuralism I wish to defend: Immanent structuralism. The "immanence" in the phrase "immanent structuralism" refers to the fact that, according to immanent structuralism, mathematics studies structural *universals* or *properties*, some of which are literally had or instantiated by physical objects, in just the way that other properties like *mass* and *charge* are. Thus, mathematical patterns, or structures, can be "located in" objects in exactly the same way that an object's size, mass, or color can. Hence, they are "immanent" to the objects that have them.[9]

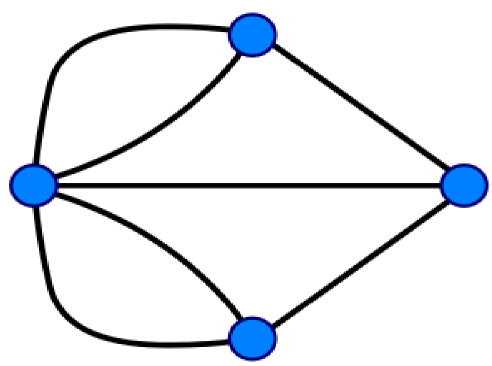
One question for the immanent structuralist is what is meant by a "structural property?" I think the clearest answer to this question comes from philosopher James Franklin:[10]

- **(PROP):** P is a purely structural property iff P can be defined entirely in terms of 'part', 'whole', 'sameness', 'difference', and purely logical vocabulary.

This is best illustrated by an example. Consider Euler's famous Bridges of Konigsberg problem:



(A): Bridge



(B): K-Graph

The question Euler set out to answer was whether there was a path through the city that would cross each bridge exactly once. (The bridges are highlighted.) However, the rules

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[9] This is in contrast to the ante rem structuralist. Cf., Resnik (1997) p. 261: "Some philosophers … have wanted to take structural properties, construed as metaphysical universals, as primitive entities and interpret mathematics within a theory of universals. … I am a realist about mathematical objects first, without being a realist about properties at all. …" See also ibid., p. 269. See also Shapiro (1997) pp. 89-90. For ante rem structuralists, a structure is more like an *exemplar* or *paradigm*, along the lines of Plato's Ideas or Forms; as such, ante rem theorists do not ultimately understand "exemplification" as straightforward property or universal-instantiation, as I would, but rather as consisting in something analogous to an isomorphism or congruence relation; see Resnik (1997) p. 204 ff. and Shapiro (1997) pp. 90-91.

[10] For this definition, see Franklin (2014) p. 57.

are that the islands can only be reached by the bridges (no swimming, flying, or wormhole-ing!) and every bridge, once accessed, must be crossed to the other side (no turning back half-way across the bridge!). One need not end up at the place one started; one only has to cross each bridge once. As it turns out, the answer to Euler's question is negative: There is no such path.

Now, what's most interesting about this case for our purposes is the fact that many of the details mentioned in the question don't *matter*, at least mathematically speaking: The question can be grasped entirely by looking at the graphical representation in (B).

I will call the type of object (B) represents a K-graph. To state the definition of a K-graph (the type of graph the question is about) all we need to mention are four distinct parts, v1, … , v4 (represented via four nodes), and some relation E between them (holding between the parts in the same way as the seven lines connect the nodes). Thus, the property of being a K-graph would seem to be a purely structural property, since it can be defined as follows:

- **(K-graph):** The property of being a K-graph is the property of being a whole G with some distinct parts v_1, … , v_4, and some relation E between these parts such that v_1Ev_2, v_1Ev_3, … (etc.).[11]

Contrast this with a paradigmatically non-purely structural property, say, being a human:

- **(Human):** The property of being an animal, and of having a rational nature, and … (etc.)

Assuming, of course, that being an animal and having a rational nature will have to be defined in irreducibly physical terms (or maybe even irreducibly mental terms), the property of being human, as defined, is not a purely structural property.

As another example, consider the Klein 4-group; it is a group with four elements, e, a, b, c, and an operation * on these elements. Its table is given below:

*	**e**	**a**	**b**	**c**
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

[11] By specifying the parts of G with variables and the relations with a predicate we are specifying the logical categories of these things, and so we can still say that this property is defined in terms of purely logical vocabulary.

We can define the *Klein 4-group* as a purely structural property:

- **(KLEIN):** The property of being a Klein 4-group is the property of being a whole that is a group[12] with distinct parts e, a, b, c, and a function[13] * on these parts such that: e*e=e, e*a=a , ... (etc.)

The property of being a Klein 4-group is spelled out by specifying the definition of * from the table we saw above. Note that many algebraic structures can be defined by similar tables.

We can also draw examples from topology and analysis. For instance, another example of a structural property is the property of being a *topological space*:[14]

- **(TOP):** The property of being a topological space is the property of being a whole S with two parts, O and C (called the open parts and closed parts), such that:
  1. There is some part e (called the empty part) that has no parts.
  2. S is a part of O and e is a part of O.
  3. Any sum of parts of O is a part of O
  4. Any finite intersection of parts of O is a part of O.

This important definition from topology allows us to give the definition of *continuity* as well:

- **(CONT):** The property of being a continuous function is the property of being some function f from parts of a topological space S to parts of S, such that the inverse image of any open part is also open.[15]

Immanent structuralism holds that, given the abstract nature of mathematics, all of the structures that mathematics studies can be defined as purely structural properties similar to these.[16] Thus, the subject matter of mathematics consists only in purely

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[12] The property of a group is itself purely structural; if one looks at a definition of a group, one will see that it just specifies some whole with some operation obeying closure, associativity, etc., where these properties are themselves definable purely structurally in terms of part, whole and logical vocabulary.

[13] Talk of functions can be reduced to talk of relations, if one finds talk in terms of relations preferable.

[14] See Franklin (2014).

[15] Hopefully it is clear how 'inverse image' would be defined too. And again, if it is easier to think in terms of relations, function talk can be wholly defined in terms of relations.

[16] For further examples, drawn from the higher reaches of mathematics, along with discussion of the role part and whole thinking plays in mathematics, see Bell (2004). I have chosen not to use Bell's examples, not because I think they don't work, but because the concept of a purely structural property is most easily seen via simple cases. Nevertheless, Bell's examples confirm that this is not just a feature of "simple" mathematics; it is the defining feature of mathematics.

structural *properties*, and does not include any Platonism-style mathematical *objects*. I will try to argue for this more fully in Chapter 4, but for now hopefully these examples are illustrative of the idea.

1.4. Immanent Structuralism: Ontology and Epistemology

One of the primary benefits I claim for immanent structuralism is that we can avoid some of the epistemological and ontological concerns that arise for Platonists and ante rem structuralists.

In the first place, we are able to get rid of sui generis Platonic mathematical objects, and only have to deal with properties (which there will be independent metaphysical pressure, from the empirical world, to deal with anyway). Thus, the categories of being required for mathematical truth are reduced.

Secondly, on my view, since mathematical properties are purely structural properties, they can be literally had or instantiated by physical objects, just as other properties like *mass*, *charge* and *color* are.[17] Thus, mathematical patterns, or structures, can be "located in" objects or regions of space in exactly the same way that an object's size, mass, or color can be. Hence, at least in principle, *some* mathematical properties and relations can be directly accessed by perception (often, though not only, by vision).

In order for claims about *uninstantiated* mathematical structures to come out true, however, we will need there to be some uninstantiated – and thus, unperceived – properties.[18] Nevertheless, if immanent structuralism is correct, all of the uninstantiated properties of mathematics can be built up *out of* directly perceivable ones – just as the property *golden mountain* can be built out of the perceivable properties *golden* and *mountain*. Thus, immanent structuralism has the potential to provide a more realistic epistemology for higher mathematics, where our initial acquisition of mathematical concepts is similar to our acquisition of concepts of other physical properties (viz., through perception); adult humans can then go on to build and define the more complicated concepts of higher mathematics via their logical concepts.

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[17] Though in virtue of their being "purely structural" properties, they can also be had by not-obviously-physical things too – e.g., I can count ideas, relations, or even angels if there are any.

[18] Though, in a way, immanent structuralism's ontology is even less committal than this: If one is ultimately a nominalist about properties, then presumably one has a way to effectively translate *all* property-talk – including talk about uninstantiated properties – into language that doesn't require reference to properties. That would be fine with me, so long as this paraphrastic elimination is able to capture all the facts about properties that I will need later.

In these respects, immanent structuralism can hope to provide a more plausible epistemology than standard Platonistic theories. Part of the problem for Platonism has been the explicitly axiomatic model that it takes as its paradigm case: Mathematical theorems are justified by appeal to some more fundamental axioms and well-defined rules of inference that are taken to be valid. Thus, the quest has been to look for the "foundational" axioms which also specify the fundamental "entities" or "objects" of mathematics. This leads to the further problem of how these foundational axioms and their accompanying postulation of mathematical objects can be justified. Various proposals have been given for how to do this, including appeals to a quasi-perceptual, direct intuition; inference to the best explanation; indispensability arguments; and revised concepts of analyticity.

Undoubtedly, advances in the axiomatic method have contributed decisively to the rigor of mathematics. However, in practice, the axiomatic model is not the primary means by which mathematical understanding is cultivated or how mathematical results are discovered; (informal) proof is very much the last step. It is also not the only way that mathematical results can be justified. And it is almost certainly not how mathematical concepts are initially acquired.

Important recent work in cognitive science and philosophy of psychology, such as Carey (2011) and Burge (2010), have argued that mathematical concepts and basic mathematical beliefs are present from an early age, and are likely even represented in the perceptual systems of sufficiently sophisticated animals. And obviously young children learn important mathematical results in grade school, and are taught by non-axiomatic, suggestive methods (with heavy emphasis on perceptual aids). Presumably these mathematical results are *known*.

So, while perhaps rigorous proof provides the *best* justification for mathematical beliefs, rigorous proof is not *necessary* for that. Thus, even if Platonists were successful in the project of providing adequate "foundations" in the form of axioms plus a plausible story of how these axioms can be known (say, via Frege's view that they are known qua analytic truths, or via some indispensability argument), this is still an implausible explanation for the vast majority of mathematical knowledge had by most people in most times and places. On immanent structuralism, on the other hand, explaining this knowledge is not more difficult than explaining how people can gain knowledge through perception and the concepts built up from perception.

1.5 A Preview of Things to Come

The work of my dissertation will be to spell out in detail this already intuitive picture and apply it to some specific phenomena. The dissertation is roughly in two parts: In the first part, Chapters 2 and 3 constitute the bulk of the theory and try to give a compelling story about the two central issues for any philosophy of mathematics: (a) ontology and truth in mathematics and (b) the epistemology of mathematics. Chapter 2 presents an essence-based account of mathematical truth that assumes the existence of Aristotelian universals. In this chapter I draw on recent work in neo-Aristotelian metaphysics that has been unavailable or underappreciated in previous discussions of mathematics. This chapter also develops and advances the metaphysics of property parthood.[19] Chapter 3 takes the account of mathematical truth and ontology in Chapter 2 and attempts to show how a plausible epistemology of mathematics falls out of this account, while clarifying further a few aspects of the ontology.

The second part of the dissertation applies the metaphysical and epistemological theory set out in Part I to more specific issues in mathematical practice and in the philosophy of science. Chapter 4 considers the practice of mathematical reduction and what I call "treating-as," and illustrates how immanent structuralism is ideally situated to explain these phenomena. Having seen the theory put to use a bit, Chapter 5 constitutes a brief interlude, and compares immanent structuralism with some closely related positions – including Shapiro and Resnik's *ante rem* structuralism, Hellman's modal structuralism, and Balaguer's modified "full-blooded" Platonism – showing how immanent structuralism avoids some of the significant pitfalls that even these more sophisticated treatments fall into.

Chapter 6 considers the notion of "*de re* mathematical necessity," which is often confused with other notions of necessity related to mathematics. I argue that immanent structuralism is *far* better placed to explain cases of de re mathematical necessity than is traditional Platonism; and if anything, perhaps surprisingly, these cases constitute a significant and underappreciated problem for Platonism. At the end I connect this discussion to recent work in mathematical explanations of physical and concrete phenomena. Chapter 7 considers inductive and non-deductive reasoning in mathematics and argues that immanent structuralism possesses some advantage in explaining why these methods work. Chapter 8 is a concluding chapter, where I attempt to draw some broader lessons for ontology and philosophical theorizing. In particular, I identify a few paradigmatic forms of reasoning that are common among analytic philosophers and ontologists, but are undercut by the theory presented here.

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[19] A notion that has seen a resurgence very recently and is likely to become increasingly important as intensionalizing accounts of semantics gain traction. See especially Craig Warmke (2015), (2016), and (2019), as well as L.A. Paul (2002), (2004) and (2012).