Immanent Structuralism Talk 10/14/17

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Part One: Background

I. Mathematical Platonism

Reductive and Sui Generis

- Platonism about a category of mathematical objects (the **natural numbers**, say) can be ***reductive*** – e.g., a certain sort of naive set-theoretic Platonism – or ***sui generis***.
- Any mathematical Platonism worthy of the name takes at least *some* "abstract" mathematical individuals to be irreducible to concrete ones.

Some Basic Worries

- **Epistemology**: How can most people have reliable beliefs about causally effete abstract objects?
- **Ontology**: The fewer the fundamental kinds of entities one posits, the better. So a theory that can plausibly amend mathematical ontology to some already-recognized category is preferable.
- **Applicability**: How can we learn things about the **physical** world by mathematics if mathematics is about a realm of **non-physical** entities? (And not just in physics, but in all *sorts* of natural sciences.)

II. Immanent Structuralism or I-Structuralism

I-Structuralism: Immanent, not Particular

- Certain forms of structuralism (e.g., Resnik's, perhaps Shapiro's) posit an ontology of intrinsically featureless "positions" whose features (natures?) are "wholly exhausted" by their relations to other positions within a "structure" of such positions.
- Between this view and mine there is **a great chasm.**
- **I-Structuralism**: Mathematics studies a particular class of **universals** or **properties** – viz., the **purely structural properties** – and these can be literally had by all kinds of different things
- Arguably, epistemology, ontology, and applicability are straightforward on this view.
- **(STRUC):** P is a **purely structural property** iff P can be defined entirely in terms of 'part', 'whole', 'sameness', 'difference', and purely logical vocabulary. (See James Franklin's work.)
- We will consider some examples later on, below. Our concern now will be to argue that I-Structuralism best accounts for mathematical reduction, treating-as, and abstraction.

Part Two: Reduction, Treating-As, and Abstraction

III. Mathematical Reduction

V-Sets and Z-Sets

- **(V-Sets)**: 0: { } 1:{ **{ }** } 2: { **{}** , {{}} } …

  In general: (i) We define the 0^{th} V-set as the empty set; (ii) Given the nth V-Set, the (n+1)-th V-set is the power set of the nth V-set, i.e. the set of all subsets of the *previous* V-set.
- **(Z-Sets):** 0: {} 1: {**{}**} 2: {{{}}} …

  In general: (i) We define the 0^{th} Z-set as the empty set; (ii) Given the nth Z-set **S**, the (n+1)-th Z-set is the set **{S}**.
- A discrete mathematics course might show one how to reduce the natural numbers.

Multiple Reductions: Against Naïve Set-Theoretic Platonism

- Is this "defining" or "reducing" ontological? A certain Naïve Set-Theoretic Platonist says **yes**.
- **Benacerraf's Worry**: There are (infinitely) many equally good ways to do the reduction. And there seems to be no principled way to choose between them. Yet if one takes mathematical reduction to be ontological, one has no choice but to choose.

Multiple Reductions: In Favor of Immanent Structuralism

- Earlier forms of structuralism make sense of this case by taking the natural numbers to just be "positions" in a "structure."
- I-Structuralism makes sense of this by taking number-theoretic statements to be about the natural number **pattern** as a whole (a **universal** that is literally instantiated by different series of sets).
- **<u>Thesis</u>**: Mathematical reduction is an instance of a more general phenomenon I will call **mathematical treating-as**. Arguably this phenomenon is best made sense of via I-Structuralism.

IV. Mathematical Treating-As

Mathematical Treating-As

- Recall the following sort of locution from one's undergraduate mathematics textbooks. It may have even been used in that discrete mathematics course while defining the natural numbers via set theory.
- **(S1):** You can think of vectors as directed line segments on a plane.

  (S2): You can think of standard propositional logic as an algebraic structure, viz., a Boolean lattice.

  (S3): You can think of the set S as a function $\mathbf{f}_s$, which assigns 1 to any object in S, 0 otherwise.

  (S4): You can think of an integral as the area under a curve.

  (S5): You can think of complex numbers as rotations around the origin of a plane.

(S6): You can think of complex numbers as ordered pairs of real numbers.

- I will call this phenomenon mathematical **treating-as** (probably "thinking-of" would work too).
- Treating-as does not carry heavy ontological implications; in each case, where you can treat X as Y, it does not seem plausible to literally *identify* the X's with the Y's in any metaphysically significant sense.
- For instance, it would not be plausible to literally identify functions, in general, with sets, and simultaneously identify sets, in general, with characteristic functions (*a la* **(S3)**).

Properties and Functions of Treating-As

- **Properties of Treating-As**: Treating-as can be **(A)** Symmetric – **(S3)**; **(B)** Non-Unique – **(V-Sets)** and **(Z-Sets)**; **(C)** Obscurum per Obscurius – **(S1)**; **(S4)**; **(S5).**
- Since it has these properties, treating-as, by itself, cannot imply ontological reductive identification.
- **Functions of Treating-As**: **(A)** *Heuristic* – Treating one entity as another sort often helps students; **(B)** *House-Cleaning* – We can show how one part of mathematics X (maybe one that has been considered dubious for some time) is no more problematic than another part of mathematics Y, either in virtue of the well-definedness of the Y's or in virtue of the assumed consistency of Y. So mathematical treating-as can help us prove (i) *well-definedness* and (ii) *relative consistency*; **(C)** *Epistemic* –For instance, if we reduce one part of mathematical discourse X to another part Y, this might give us clues as to how we should axiomatize X, since we may be able to catch onto a pattern that is more clear or evident in Y, or we may be able to see that certain proposed axioms are redundant, unnecessary, or maybe don't even accurately describe the relevant parts of Y.

Mathematical Reduction as Mathematical Treating-As

- **Properties of Mathematical Reductions**: Asymmetry; Non-Uniqueness
- **Functions of Mathematical Reductions**: **Heuristic, House-Cleaning, Epistemic**
- **Reductions vs. Treating-As Generally**: In the case of mathematical reduction, we treat an entire system that we are concerned about as being part of another system whose "credentials" are already firmly established. (The complex numbers come to mind.) Generally, total definability is necessary.
- If we think of reduction as an instance of treating-as, it becomes clear why it can be non-unique.
- Admittedly, mathematical reduction is asymmetric. But the asymmetry in mathematical reductions is an **epistemic asymmetry**, induced by considerations of **rigor**. (Ignotum per ignotius.) This is *predicted* by my view.

I-Structuralism and Treating-As

- **(TREAT):** We can treat a mathematical structure X as Y if and only if X is instantiated by Y.
- "Treat" in the strict sense that is. Maybe sometimes, strictly speaking, we only treat one *part* of a structure as something else (especially when our purposes are heuristic).

- If **(TREAT)** is right, treating-as will have all of the properties we expected, as above.
- I-Structuralism also predicts that treating-as and "reduction" will not be ontologically significant.

V. Mathematical Abstraction

- Note a certain weakness in the condition **(TREAT)** above.
- **Mathematics Professor**: "Treat it as whatever you like!"
- **Philosopher**: Well, not ***whatever*** you like; but anything you like that instantiates the structure.
- **(ABS)**: For the purposes of mathematical truth, it does not matter *what* you take the names in mathematics to refer to, so long as you can treat the things referred to as such.
- **Examples: V-Sets and Z-Sets, Klein 4-Groups, K-Graphs**

VI. I-Structuralism and Structural Properties: Examples

- **(STRUC):** P is a **purely structural property** iff P can be defined entirely in terms of 'part', 'whole', 'sameness', 'difference', and purely logical vocabulary.
- **(K-graph):** The property of being a **K-graph** is the property of being a whole G with some distinct parts $v_1, \ldots, v_4$, and some relation E between these parts such that v_1Ev_2, v_1Ev_3, … (etc.).
- **(Human):** The property of being an animal, and of having a rational nature, and … (etc.)
- **(KLEIN):** The property of being a Klein 4-group is the property of being a whole that is a group with distinct parts e, a, b, c, and a function * on these parts such that: $e*e=e$, $e*a=a$, ... (etc.)
- **(TOP):** The property of being a **topological space** is the property of being a whole S with two collections of parts, O and C (called the open parts and closed parts), such that:
  1. There is some part of S, e (called the empty part), that has no parts.
  2. S is in O and e is in O.
  3. Any sum of parts in O is in O
  4. Any finite intersection of parts in O is in O.
- **(CONT):** The property of being a **continuous function** is the property of being a function f from parts of a topological space S to parts of S, such that the inverse image of any open part is also open.
- **Abbreviations**: **Σx: -** "For some x such that"; **Σ!x -** "For exactly one x"; **ix: -** "The x such that"; **Πx: -** "For all x such that"; **x∘S -**"x is a part of S"; **U≪S -** "U is a whole which is a part of S or = S"
- **(PA):** The property of instantiating Peano Arithmetic is the property of being a whole S with a function ' from parts of S to parts of S such that:

    (1) $\Sigma!x\circ S{:}\ \sim\Sigma y\circ S{:}\ x=y'$; $0 \stackrel{\text{def}}{=} ix\circ S{:}\ \sim\Sigma y\circ S{:}\ x=y'$

    (2) $\Pi x{:}\ (x\circ S \Rightarrow x'\circ S)$

    (3) $\Pi x{:}\ \Pi y{:}\ (x'=y' \Rightarrow x=y)$

    (4) $\Pi U{:}\ U\ll S \Rightarrow [[0\circ U \ \&\ \Pi x\circ S{:}\ (x\circ U\Rightarrow x'\circ U)] \Rightarrow \Pi x\circ S{:}\ x\circ U]$

- I-Structuralism handles these cases nicely. And these are not the only cases (see John Bell's work).